

Hopf surfaces: a family of locally conformal Kähler metrics and elliptic fibrations

Maurizio Parton

Abstract

In this paper we describe a family of locally conformal Kähler metrics on class 1 Hopf surfaces $H_{\alpha,\beta}$ containing some recent metrics constructed in [GO98]. We study some canonical foliations associated to these metrics, in particular a 2-dimensional foliation $\mathcal{E}_{\alpha,\beta}$ that is shown to be independent of the metric. We elementary prove that $\mathcal{E}_{\alpha,\beta}$ has compact leaves if and only if $\alpha^m = \beta^n$ for some integers m and n , namely in the elliptic case. In this case the leaves of $\mathcal{E}_{\alpha,\beta}$ give explicitly the elliptic fibration of $H_{\alpha,\beta}$, and the natural orbifold structure on the leaf space is illustrated.

1 Introduction

The study of metrics on complex surfaces arose in the sixties out of Kodaira's classification of minimal complex surfaces in seven classes I_0, \dots, VII_0 (see [Kod64, Kod66, Kod68a, Kod68b]): which complex surfaces, with respect to this classification, admit a Kählerian metric? The surfaces in classes I_0 , III_0 and V_0 are easily seen to be Kähler, while the surfaces in classes VI_0 and VII_0 are not, due to topological obstructions (their first Betti number is odd). The surfaces in class IV_0 are Kähler as shown by Miyaoka and in 1983, when Todorov and Siu proved that every surface of class II_0 is Kähler, the question was at last settled: only the surfaces of classes VI_0 and VII_0 are not Kähler (see for instance [BPV84]).

Is there a weakened version of the Kähler hypothesis that we can hope to prove for surfaces in classes VI_0 and VII_0 ? The notion of locally conformal Kähler manifold was introduced in this context by I. Vaisman in [Vai76]; in [Vai79] he thoroughly studied locally conformal Kähler metrics with parallel Lee form; subsequently F. Tricerri in [Tri82] gave an example of a locally conformal Kähler metric with non-parallel Lee form. Further properties of locally conformal Kähler manifolds were proved by B. Y. Chen and P. Piccinni in [CP85]; in particular, the existence on them of some canonical foliations. Until 1998, there were very few examples of locally conformal Kähler manifolds, namely *some* Hopf surfaces, some Inoue surfaces and manifolds of type $(G/\Lambda) \times S^1$ where G is a nilpotent or solvable group. Recent results were obtained by P. Gauduchon and L. Ornea in the paper

[GO98], where they showed that *every* primary Hopf surface is locally conformal Kähler by finding a (family of) locally conformal Kähler metric (with parallel Lee form) on those of class 1 and then deforming it; and by F. A. Belgun in [Bel99] where he classified the locally conformal Kähler surfaces with parallel Lee form and showed that also secondary Hopf surfaces are locally conformal Kähler.

In this paper we show that the metrics written in [GO98] for Hopf surfaces of class 1 belong to a family of locally conformal Kähler metrics that are parametrized by the smooth positive functions defined on the circle S^1 . Among all these locally conformal Kähler metrics, the only ones with parallel Lee form are those of [GO98]. Then we explicitly study the canonical foliations associated to the metrics of this family. Class 1 Hopf surfaces $H_{\alpha,\beta}$ are elliptic if and only if an algebraic condition is satisfied, that is $\alpha^n = \beta^m$ for some integers n and m (see [Kod64, 2]). We find that, whenever this condition is satisfied, one of the canonical foliations gives exactly the elliptic fibration. Finally, we examine the regularity of this foliation and the natural orbifold structure on the leaf space.

In section 2 we give some basic preliminaries and we enounce more precisely the theorem of [GO98] we used (see theorem 2.2).

In section 3.1 we develop some tools we shall need, namely a diffeomorphism between $H_{\alpha,\beta}$ and $S^1 \times S^3$ (see formula (5)), a parallelization on $S^1 \times S^3$ (see formulas (6)) and the explicit description of the induced complex structure on $S^1 \times S^3$ via the diffeomorphism (see formulas (8)). This is the point of view we adopt to study $H_{\alpha,\beta}$.

In section 3.2 we study the simplest case, that is $\alpha = \beta$: we note that with our point of view there is a nice interpretation of the classical invariant metric $(dz_1 \otimes d\bar{z}_1 + dz_2 \otimes d\bar{z}_2)/(z_1\bar{z}_1 + z_2\bar{z}_2)$ on $\mathbb{C}^2 - 0$ (it is expressed as the identity matrix), then we deform it by means of a positive function $h: S^1 \rightarrow \mathbb{R}$ and we obtain a family of locally conformal Kähler metrics. The Lee form is parallel if and only if h is constant.

In section 3.3 we generalize the previous results to the case $\|\alpha\| = \|\beta\|$.

In section 3.4 we observe that the previous cases do not generalize directly, since there is no classical invariant metric in this case. We apply our method to the metric of [GO98] to obtain a family of locally conformal Kähler metrics on $H_{\alpha,\beta}$ (see theorem 3.3) parametrized by the real positive functions on S^1 . Then we verify that the only metrics with parallel Lee form in this family are the ones of [GO98] (see theorem 3.3).

In section 4 we begin by recalling the definitions of four canonical distributions on a locally conformal Kähler manifold, as given in [CP85]⁽¹⁾, then we study each of them in detail. We remark that they are all integrable and explicitly find the leaves, then we study their properties obtaining necessary and sufficient conditions for compactness (see theorems 4.2, 4.3 and 4.6).

⁽¹⁾In [CP85, proposition 4.7] these distributions are studied for diagonal Hopf surfaces H_α with the metric $(dz_1 \otimes d\bar{z}_1 + dz_2 \otimes d\bar{z}_2)/(z_1\bar{z}_1 + z_2\bar{z}_2)$. Other canonical distributions are studied in [Pic90].

In section 5 we recall the definition of elliptic surface, as given in [Kod64]. Then we show that when the foliation $\mathcal{E}_{\alpha,\beta}$ has all compact leaves -and this happens, according to theorem 4.6, if and only if $\alpha^m = \beta^n$ for some integers n and m -, we can identify the leaf space with $\mathbb{P}^1\mathbb{C}$ in such a way that the canonical projection is a holomorphic map (see theorem 5.1). This means that, whenever $H_{\alpha,\beta}$ is elliptic, the ellipticity is explicitly given by the foliation $\mathcal{E}_{\alpha,\beta}$.

In section 6 we recall the definitions of regularity and quasi-regularity, and we show that $\mathcal{E}_{\alpha,\beta}$ is quasi-regular if and only if $H_{\alpha,\beta}$ is elliptic, and it is regular if and only if $\alpha = \beta$. The quasi-regularity gives the leaf space a natural structure of orbifold with two conical points.

2 Preliminaries

A Hermitian manifold (M^{2n}, J, g) is called *locally conformal Kähler*, briefly *l.c.K.*, if there exist an open covering $\{U_i\}_{i \in I}$ of M and a family $\{f_i\}_{i \in I}$ of smooth functions $f_i: U_i \rightarrow \mathbb{R}$ such that the metrics g_i on U_i given by

$$g_i \stackrel{\text{def}}{=} e^{-f_i} g|_{U_i}$$

are Kählerian metrics. The following relation holds on U_i between the fundamental forms Ω_i and $\Omega|_{U_i}$ respectively of g_i and $g|_{U_i}$:

$$\Omega_i = e^{-f_i} \Omega|_{U_i},$$

so the *Lee form* ω locally defined by

$$\omega|_{U_i} \stackrel{\text{def}}{=} df_i \tag{1}$$

is in fact global, and satisfies $d\Omega = \omega \wedge \Omega$. The manifold (M, J, g) is then l.c.K. if and only if there exists a global closed 1-form ω such that

$$d\Omega = \omega \wedge \Omega$$

(see for instance the recent book [DO98]).

As Kodaira defined in [Kod66, 10], a *Hopf surface* is a complex compact surface H whose universal covering is $\mathbb{C}^2 - 0$. If $\pi_1(H) \simeq \mathbb{Z}$ then we say that H is a *primary Hopf surface*. Kodaira showed that every primary Hopf surface can be obtained as

$$\frac{\mathbb{C}^2 - 0}{< f >}, \quad f(z_1, z_2) \stackrel{\text{def}}{=} (\alpha z_1 + \lambda z_2^m, \beta z_2),$$

where m is a positive integer and α, β and λ are complex numbers such that

$$(\alpha - \beta^m)\lambda = 0 \quad \text{and} \quad \|\alpha\| \geq \|\beta\| > 1.$$

We write $H_{\alpha,\beta,\lambda,m}$ for the generic primary Hopf surface. If $\lambda \neq 0$ we have

$$f(z_1, z_2) = (\beta^m z_1 + \lambda z_2^m, \beta z_2)$$

and the surface $H_{\beta,\lambda,m} \stackrel{\text{def}}{=} H_{\beta^m, \beta, \lambda, m}$ is called *of class 0*, while if $\lambda = 0$ we have

$$f(z_1, z_2) = (\alpha z_1, \beta z_2)$$

and the surface $H_{\alpha,\beta} \stackrel{\text{def}}{=} H_{\alpha,\beta,0,m}$ is called *of class 1* (this terminology refers to the notion of *Kähler rank* as given in [HL83, § 9]).

A globally conformal Kähler metric on $\mathbb{C}^2 - 0$ (that is, of the form $e^{-f}g$ where $f: \mathbb{C}^2 - 0 \rightarrow \mathbb{R}$ and g is Kähler), which is invariant for the map $(z_1, z_2) \mapsto (\alpha z_1 + \lambda z_2^m, \beta z_2)$, defines a l.c.K. metric on $H_{\alpha,\beta,\lambda,m}$: this is the case for the metric

$$\frac{dz_1 \otimes d\bar{z}_1 + dz_2 \otimes d\bar{z}_2}{z_1 \bar{z}_1 + z_2 \bar{z}_2} \quad (2)$$

which is invariant for the map $(z_1, z_2) \mapsto (\alpha z_1, \beta z_2)$ (and so defines a l.c.K. metric on $H_{\alpha,\beta}$) whenever $\|\alpha\| = \|\beta\|$. The Lee form of this metric is parallel for the Levi-Civita connection (see [Vai79]).

In [Vai82], I. Vaisman called *generalized Hopf (g.H.)* manifolds those l.c.K. manifolds (M, J, g) with a parallel Lee form. Recently, since F. A. Belgun proved that primary Hopf surfaces of class 0 do not admit any generalized Hopf structure (see [Bel99]), some authors (see for instance [DO98, GO98]) decided to use the term *Vaisman manifold* instead. We shall adhere to this terminology and thus give the following

Definition 2.1 A *Vaisman manifold* is a l.c.K. manifold (M, J, g) with parallel Lee form with respect to the Levi-Civita connection of g .

Define the operator d^c by $d^c(f)(X) \stackrel{\text{def}}{=} -df(J(X))$ for $f \in C^\infty$ and $X \in \mathfrak{X}(M)$, and call *potential* on the open set \mathcal{U} of the complex manifold (M, J) a map $f: \mathcal{U} \rightarrow \mathbb{R}$ such that the 2-form on \mathcal{U} of type $(1, 1)$ given by $(dd^c f)/2$ is positive: namely, such that the bilinear map g on $\mathfrak{X}(\mathcal{U}) \times \mathfrak{X}(\mathcal{U})$ given by

$$g(X, Y) \stackrel{\text{def}}{=} -\frac{dd^c f}{2}(J(X), Y)$$

is a (Kählerian) metric on \mathcal{U} .

Take the potential $\Phi_{\alpha,\beta}: \mathbb{C}^2 - 0 \rightarrow \mathbb{R}$ given by

$$\Phi_{\alpha,\beta}(z_1, z_2) \stackrel{\text{def}}{=} e^{\frac{(\log \|\alpha\| + \log \|\beta\|)\theta}{2\pi}} \quad (3)$$

where θ is given by

$$\frac{\|z_1\|^2}{e^{\frac{\theta \log \|\alpha\|}{\pi}}} + \frac{\|z_2\|^2}{e^{\frac{\theta \log \|\beta\|}{\pi}}} = 1. \quad (4)$$

In [GO98] the following theorem is proved:

Theorem 2.2 ([GO98, Proposition 1 and Corollary 1]) *The metric associated to the 2-form of type $(1, 1)$ on $\mathbb{C}^2 - 0$*

$$\frac{dd^c \Phi_{\alpha,\beta}}{2\Phi_{\alpha,\beta}}$$

is invariant for the map $(z_1, z_2) \mapsto (\alpha z_1, \beta z_2)$. The induced metric on $H_{\alpha,\beta}$ is Vaisman for every α and β .

3 Some metrics on $S^1 \times S^3$

3.1 Definitions, notations and preliminary tools

We look at the 3-sphere as

$$S^3 \stackrel{\text{def}}{=} \{(\xi_1, \xi_2) \in \mathbb{C}^2 : \|\xi_1\|^2 + \|\xi_2\|^2 = 1\}$$

and at S^1 as the quotient of \mathbb{R} by the map $\theta \mapsto \theta + 2\pi$. The manifolds $S^1 \times S^3$ and $H_{\alpha,\beta}$ are diffeomorphic (see [Kat75, theorem 9]) by means of the map $F_{\alpha,\beta}$ given by F in the diagram

$$\begin{array}{ccc} \mathbb{R} \times S^3 & \xrightarrow{F} & \mathbb{C}^2 - 0 \\ h \downarrow & & \downarrow f \\ \mathbb{R} \times S^3 & \xrightarrow{F} & \mathbb{C}^2 - 0 \end{array}$$

where

$$\begin{aligned} h(\theta, (\xi_1, \xi_2)) &\stackrel{\text{def}}{=} (\theta + 2\pi, (\xi_1, \xi_2)), \\ f(\xi_1, \xi_2) &\stackrel{\text{def}}{=} (\alpha \xi_1, \beta \xi_2), \\ F(\theta, (\xi_1, \xi_2)) &\stackrel{\text{def}}{=} (e^{\frac{\theta \log \alpha}{2\pi}} \xi_1, e^{\frac{\theta \log \beta}{2\pi}} \xi_2). \end{aligned}$$

If $[z_1, z_2]$ is the element in $H_{\alpha,\beta}$ corresponding to $(z_1, z_2) \in \mathbb{C}^2 - 0$, we have

$$F_{\alpha,\beta}(\theta, (\xi_1, \xi_2)) \stackrel{\text{def}}{=} [e^{\frac{\theta \log \alpha}{2\pi}} \xi_1, e^{\frac{\theta \log \beta}{2\pi}} \xi_2] \quad (5)$$

and the inverse is

$$F_{\alpha,\beta}^{-1}([z_1, z_2]) = (\theta, (e^{-\frac{\theta \log \alpha}{2\pi}} z_1, e^{-\frac{\theta \log \beta}{2\pi}} z_2))$$

where θ is given by (4).

Via this diffeomorphism we can transfer the complex structure of $H_{\alpha,\beta}$ to $S^1 \times S^3$. We shall use the notation $J_{\alpha,\beta}$ for this complex structure on $S^1 \times S^3$; in particular the $J_{\alpha,\alpha}$ on $S^1 \times S^3$ were studied and classified by P. Gauduchon in [Gau81, propositions 2 and 3, pages 138 and 140], by means of the parallelizability of $S^1 \times S^3$.

Let \mathbb{H} be the non commutative field of quaternions, and let us identify it with \mathbb{C}^2 by means of $(\xi_1, \xi_2) \mapsto \xi_1 + j\xi_2$. Let θ be the point in $S^1 \subset \mathbb{C}$ given by the embedding $\theta \mapsto e^{i\theta}$, and let $Q = Q(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ be the point in $S^3 \subset \mathbb{H}$ that, in the above identification, gives the complex numbers $\xi_1 = \alpha_1 + i\alpha_2$ and $\xi_2 = \alpha_3 + i\alpha_4$. We shall use the parallelization $\mathcal{E} \stackrel{\text{def}}{=} (e_1, e_2, e_3, e_4)$ on $S^1 \times S^3$ (and its dual $\mathcal{E}^* = (e^1, e^2, e^3, e^4)$):

$$\begin{aligned} e_1((\theta, Q)) &\stackrel{\text{def}}{=} ie^{i\theta} \in T_\theta(S^1), \\ e_2((\theta, Q)) &\stackrel{\text{def}}{=} iQ = (i\xi_1, i\xi_2) = (-\alpha_2, \alpha_1, -\alpha_4, \alpha_3) \in T_Q(S^3), \\ e_3((\theta, Q)) &\stackrel{\text{def}}{=} jQ = (-\bar{\xi}_2, \bar{\xi}_1) = (-\alpha_3, \alpha_4, \alpha_1, -\alpha_2) \in T_Q(S^3), \\ e_4((\theta, Q)) &\stackrel{\text{def}}{=} kQ = (-i\bar{\xi}_2, i\bar{\xi}_1) = (-\alpha_4, -\alpha_3, \alpha_2, \alpha_1) \in T_Q(S^3). \end{aligned} \tag{6}$$

The differential structure of this frame is given by the following formulas:

$$de^1 = 0, \quad de^2 = 2e^3 \wedge e^4, \quad de^3 = -2e^2 \wedge e^4, \quad de^4 = 2e^2 \wedge e^3,$$

and the non-zero brackets are

$$[e_2, e_3] = -2e_4, \quad [e_2, e_4] = 2e_3, \quad [e_3, e_4] = -2e_2.$$

One finds that

$$dF = \left(\frac{\log \alpha}{2\pi} e^{\frac{\theta \log \alpha}{2\pi}} \xi_1 d\theta + e^{\frac{\theta \log \alpha}{2\pi}} d\xi_1 \right) \otimes \partial_{z_1} + \left(\frac{\log \beta}{2\pi} e^{\frac{\theta \log \beta}{2\pi}} \xi_2 d\theta + e^{\frac{\theta \log \beta}{2\pi}} d\xi_2 \right) \otimes \partial_{z_2}. \tag{7}$$

Letting G be the complex function on $S^1 \times S^3$ given by (see [GO98, formula 45])

$$\begin{aligned} G(\theta, (\xi_1, \xi_2)) &\stackrel{\text{def}}{=} \|\xi_1\|^2 \log \alpha + \|\xi_2\|^2 \log \beta \\ &= \|\xi_1\|^2 \log \|\alpha\| + \|\xi_2\|^2 \log \|\beta\| + i(\|\xi_1\|^2 \arg \alpha + \|\xi_2\|^2 \arg \beta), \end{aligned}$$

the complex structure $J_{\alpha, \beta}$ with respect to the basis \mathcal{E} is given by

$$\begin{aligned} J_{\alpha, \beta}(e_1) &= -\frac{\Im G}{\Re G} e_1 + \frac{\|G\|^2}{2\pi \Re G} e_2 - \frac{\Re(i\xi_1 \xi_2 \bar{G} \log(\alpha/\beta))}{2\pi \Re G} e_3 - \frac{\Im(i\xi_1 \xi_2 \bar{G} \log(\alpha/\beta))}{2\pi \Re G} e_4, \\ J_{\alpha, \beta}(e_2) &= -\frac{2\pi}{\Re G} e_1 + \frac{\Im G}{\Re G} e_2 - \frac{\Re(\xi_1 \xi_2 \log(\alpha/\beta))}{\Re G} e_3 - \frac{\Im(\xi_1 \xi_2 \log(\alpha/\beta))}{\Re G} e_4, \\ J_{\alpha, \beta}(e_3) &= e_4, \\ J_{\alpha, \beta}(e_4) &= -e_3, \end{aligned} \tag{8}$$

(see [GO98, formulas 49], where the notations T , Z , E , iE , z_1 , z_2 and F are used instead of $2\pi e_1$, e_2 , $-e_3$, $-e_4$, ξ_1 , ξ_2 and G).

The real vector bundle $T(S^1 \times S^3)$ of rank 4 becomes a complex vector bundle of rank 2 by means of $J_{\alpha, \beta}$: the two vector fields e_2 and e_3 are independent over the complex numbers and, with respect to this basis, a hermitian metric on $S^1 \times S^3$ is expressed by means of a hermitian 2×2 matrix.

3.2 Case $\alpha = \beta$

Since the pull-back of the metric (2) by means of $F_{\alpha,\alpha}$ is the identity matrix in the $J_{\alpha,\alpha}$ -complex basis (e_2, e_3) of $T(S^1 \times S^3)$, we wonder whether there exist other l.c.K. metrics given by hermitian matrices of the form

$$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} \quad (9)$$

where $k: S^1 \times S^3 \rightarrow \mathbb{R}^+$ is any real positive function; the Lee form is given by

$$\omega = -k \frac{\log \|\alpha\|}{\pi} e^1$$

and by imposing the l.c.K. condition $d\omega = 0$ we obtain

$$e_2(k) = 0, \quad \log \|\alpha\| e_3(k) + \pi e_1 \left(\frac{e_3(k)}{k} \right) = 0, \quad \log \|\alpha\| e_4(k) + \pi e_1 \left(\frac{e_4(k)}{k} \right) = 0. \quad (10)$$

This second order differential system is certainly solved by a function k which satisfies $e_2(k) = e_3(k) = e_4(k) = 0$, namely, which depends only on θ ; using $F_{\alpha,\alpha}$ in the opposite direction we obtain the invariant metrics on $\mathbb{C}^2 - 0$:

$$\begin{aligned} (\|z_1\|^2 + \|z_2\|^2)^{-2} \left(& (k(\theta) z_1 \bar{z}_1 + z_2 \bar{z}_2) dz_1 \otimes d\bar{z}_1 + (k(\theta) - 1) z_2 \bar{z}_1 dz_1 \otimes d\bar{z}_2 \right. \\ & \left. + (k(\theta) - 1) z_1 \bar{z}_2 dz_2 \otimes d\bar{z}_1 + (z_1 \bar{z}_1 + k(\theta) z_2 \bar{z}_2) dz_2 \otimes d\bar{z}_2 \right) \end{aligned} \quad (11)$$

where

$$\theta = \frac{\log(\|z_1\|^2 + \|z_2\|^2)}{2 \log \|\alpha\|}$$

and k is a positive function on S^1 , i.e. a positive 2π -periodic real variable function.

Let us call θ_j^k the 1-forms

$$\theta_j^k \stackrel{\text{def}}{=} \sum_{i=1}^4 \Gamma_{ij}^k e^i$$

of the Levi-Civita connection. By the structure equations of Cartan we obtain

$$\begin{aligned} \theta_1^1 &= \frac{k'(\log^2 \|\alpha\| - \arg^2 \alpha)}{2k \log^2 \|\alpha\|} e^1 - \frac{\pi k' \arg \alpha}{k \log^2 \|\alpha\|} e^2, & \theta_2^1 &= -\frac{\pi k' \arg \alpha}{k \log^2 \|\alpha\|} e^1 - \frac{2\pi^2 k'}{k \log^2 \|\alpha\|} e^2, \\ \theta_1^2 &= \frac{k' \|\log \alpha\|^2 \arg \alpha}{4\pi k \log^2 \|\alpha\|} e^1 + \frac{k' \|\log \alpha\|^2}{2k \log^2 \|\alpha\|} e^2, & \theta_2^2 &= \frac{k' \|\log \alpha\|^2}{2k \log^2 \|\alpha\|} e^1 + \frac{\pi k' \arg \alpha}{k \log^2 \|\alpha\|} e^2, \\ \theta_3^2 &= e^4, & \theta_2^3 &= -ke^4, & \theta_2^4 &= ke^3, & \theta_4^2 &= -e^3, & \theta_4^3 &= -\frac{k \arg \alpha}{2\pi} e^1 + (2-k)e^2, \\ \theta_1^3 &= -\frac{k \arg \alpha}{2\pi} e^4, & \theta_1^4 &= \frac{k \arg \alpha}{2\pi} e^3, & \theta_3^4 &= \frac{k \arg \alpha}{2\pi} e^1 + (k-2)e^2, \\ \theta_3^1 &= \theta_4^1 = \theta_3^3 = \theta_4^4 = 0. \end{aligned}$$

A straightforward calculation thus gives

$$\begin{aligned}\nabla_{e_1}\omega &= -\frac{k'\|\log\alpha\|^2}{2\pi\log\|\alpha\|}e^1 - \frac{k'\arg\alpha}{\log\|\alpha\|}e^2, & \nabla_{e_2}\omega &= -\frac{k'\arg\alpha}{\log\|\alpha\|}e^1 - \frac{2\pi k'}{\log\|\alpha\|}e^2, \\ \nabla_{e_3}\omega &= \nabla_{e_4}\omega = 0.\end{aligned}$$

So, in the family of l.c.K. metrics given in (11), the Vaisman ones are those in which k is a constant function:

$$(\|z_1\|^2 + \|z_2\|^2)^{-2} \left((kz_1\bar{z}_1 + z_2\bar{z}_2)dz_1 \otimes d\bar{z}_1 + (k-1)z_2\bar{z}_1 dz_1 \otimes d\bar{z}_2 \right. \\ \left. + (k-1)z_1\bar{z}_2 dz_2 \otimes d\bar{z}_1 + (z_1\bar{z}_1 + kz_2\bar{z}_2)dz_2 \otimes d\bar{z}_2 \right).$$

3.3 Case $\|\alpha\| = \|\beta\|$

Again the pull-back via $F_{\alpha,\beta}$ of the metric (2) is given by the identity matrix in the $J_{\alpha,\beta}$ -complex basis (e_2, e_3) , and we can repeat the same construction: the hermitian matrix

$$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$$

where $k: S^1 \times S^3 \rightarrow \mathbb{R}^+$ is a real positive function, is a l.c.K. metric if and only if it is a solution of

$$\begin{aligned}e_2(k) &= 0, \\ \arg \frac{\alpha}{\beta} \left((\|\xi_1\|^2 - \|\xi_2\|^2) \frac{e_4(k)}{k} - \Im m(\xi_1\xi_2)e_3 \left(\frac{e_3(k)}{k} \right) + \Re e(\xi_1\xi_2)e_3 \left(\frac{e_4(k)}{k} \right) \right) \\ &\quad + 2 \left(\log \|\alpha\| e_3(k) + \pi e_1 \left(\frac{e_3(k)}{k} \right) \right) = 0, \\ \arg \frac{\alpha}{\beta} \left((\|\xi_1\|^2 - \|\xi_2\|^2) \frac{e_3(k)}{k} - \Re e(\xi_1\xi_2)e_4 \left(\frac{e_4(k)}{k} \right) + \Im m(\xi_1\xi_2)e_3 \left(\frac{e_4(k)}{k} \right) \right) \\ &\quad + \arg \frac{\alpha}{\beta} \Im m(\xi_1\xi_2) \frac{(\log \|\alpha\| e_4(k)k - 1)e_4(e_3(k))}{k^2} - 2 \left(\log \|\alpha\| e_4(k) + \pi e_1 \left(\frac{e_4(k)}{k} \right) \right) = 0.\end{aligned}$$

Computations are now much harder, due to the factor $\arg(\alpha/\beta)$: nevertheless we obtain again that any function $k: S^1 \subset S^1 \times S^3 \rightarrow R^+$ is a solution, and we again obtain

$$\begin{aligned}\nabla_{e_1}\omega &= -\frac{k'\|G\|^2}{2\pi\log\|\alpha\|}e^1 - \frac{k'\Im m G}{\log\|\alpha\|}e^2, & \nabla_{e_2}\omega &= -\frac{k'\Im m G}{\log\|\alpha\|}e^1 - \frac{2\pi k'}{\log\|\alpha\|}e^2, \\ \nabla_{e_3}\omega &= \nabla_{e_4}\omega = 0,\end{aligned}$$

that is, the l.c.K. metric given in the complex basis (e_2, e_3) by $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$ is a Vaisman metric if and only if k is constant. We thus get the following

Proposition 3.1 *The formula (9) gives a family of l.c.K. metrics on $H_{\alpha,\beta}$, in the case $\|\alpha\| = \|\beta\|$. In this family the Vaisman ones are given exactly by constant functions k .*

Remark 3.2 A family $\{g_t\}_{t>-1}$ of l.c.K. metrics (in the case $\|\alpha\| = \|\beta\|$) can be found in [Vai82, formula 2.13]. The metrics of this family coincide (up to coefficients) with the metrics of our family with k constant, where $k = t + 1$. The claim, on page 240 of [Vai82], that only g_0 has parallel Lee form is uncorrect. The author uses the Weyl connection with the hypothesis $\omega_t(B_t) = \|\omega_t\|^2 = 1$, before proving that ω_t is parallel: in such a way, what is in fact proved is that g_0 is the only metric with $\nabla\omega = 0$ and $\|\omega_t\| = 1$. Actually, by using (2.14) and (2.17), one can check that $\|\omega_t\| = 1 + t$, hence the same computation proves that all the g_t have parallel Lee form. I acknowledge a useful conversation and an exchange of e-mail messages with I. Vaisman.

3.4 General case

Unfortunately the same construction doesn't apply to the general case since the metric (2) is not invariant, hence is not defined on $H_{\alpha,\beta}$

As a starting point we use the l.c.K. metric given by P. Gauduchon and L. Ornea in the recent work [GO98]. At the beginning of their paper they explicitly find a family of Vaisman metrics on $H_{\alpha,\beta}$ by modifying the potential of (2): we make a further modification, using the same ideas of the previous cases.

Let $l: \mathcal{U} \rightarrow \mathbb{R}$ be a real function defined on an open set \mathcal{U} of \mathbb{R} , and

$$\Phi_{\alpha,\beta}: \frac{\mathcal{U}}{2\pi\mathbb{Z}} \times S^3 \rightarrow \mathbb{R}^+$$

the real positive function given by

$$\Phi_{\alpha,\beta}((\theta, (\xi_1, \xi_2))) \stackrel{\text{def}}{=} e^{l(\theta)}. \quad (12)$$

The local 2-form $\Omega \stackrel{\text{def}}{=} \frac{1}{2}dd^c\Phi_{\alpha,\beta}$ is

$$\begin{aligned} \Omega = & \frac{\Phi_{\alpha,\beta}\pi l'}{\Re e G} \left(\frac{l'^2 + l''}{l'} e^{12} - \frac{\Re e(\xi_1\xi_2)(\log \|\alpha\| \arg \beta - \log \|\beta\| \arg \alpha)}{\pi \Re e G} e^{13} \right. \\ & - \frac{\Im m(\xi_1\xi_2)(\log \|\alpha\| \arg \beta - \log \|\beta\| \arg \alpha)}{\pi \Re e G} e^{14} - \frac{2\Re e(\xi_1\xi_2) \log(\|\alpha\|/\|\beta\|)}{\Re e G} e^{23} \\ & \left. - \frac{2\Im m(\xi_1\xi_2) \log(\|\alpha\|/\|\beta\|)}{\Re e G} e^{24} + 2e^{34} \right) \end{aligned}$$

where we denote with e^{ij} the wedge product $e^i \wedge e^j$. The matrix of the hermitian bilinear form⁽²⁾ in the complex basis (e_2, e_3) of $S^1 \times S^3$ is

$$2\Phi_{\alpha,\beta}\pi l' A \quad (13)$$

⁽²⁾Given by $H(X, Y) \stackrel{\text{def}}{=} -\Omega(JX, Y) - i\Omega(X, Y)$.

where

$$A \stackrel{\text{def}}{=} \begin{pmatrix} \frac{\pi}{\Re e^2 G} \frac{l'^2 + l''}{l'} + \frac{\|\xi_1\|^2 \|\xi_2\|^2 \log^2(\|\alpha\|/\|\beta\|)}{\Re e^3 G} & \frac{i\xi_1 \xi_2 \log(\|\alpha\|/\|\beta\|)}{\Re e^2 G} \\ \frac{i\xi_1 \xi_2 \log(\|\alpha\|/\|\beta\|)}{\Re e^2 G} & \frac{1}{\Re e G} \end{pmatrix}$$

The condition that Ω be positive translates then in l' and $l'^2 + l''$ both positive. So we have a local generalization of the proposition 1 in [GO98], that is we can take the local function e^l as a potential $\Phi_{\alpha,\beta}$, where l is increasing and $l'^2 + l'' > 0$ on \mathcal{U} .

In the matrix A the dependance on θ is only given by $(l'^2 + l'')/l'$. Consider a family $\{l_{\mathcal{U}}\}_{\mathcal{U} \in U}$ of local functions, where U is an open covering of \mathbb{R} , all satisfying $l' > 0$ and $l'^2 + l'' > 0$ and such that the quantities $(l'^2 + l'')/l'$ paste to a well defined function h on S^1 . The matrix (13) then gives a global hermitian l.c.K. metric on $(S^1 \times S^3, J_{\alpha,\beta})$. In fact such a family can be found, as we show in the following

Theorem 3.3 *Given any real positive function h with period 2π on \mathbb{R} , the metric $g_{\alpha,\beta}^h$ given in the complex basis (e_2, e_3) of $T(S^1 \times S^3)$ by the hermitian matrix*

$$\begin{pmatrix} \frac{\pi h}{\Re e^2 G} + \frac{\|\xi_1\|^2 \|\xi_2\|^2 \log^2(\|\alpha\|/\|\beta\|)}{\Re e^3 G} & \frac{i\xi_1 \xi_2 \log(\|\alpha\|/\|\beta\|)}{\Re e^2 G} \\ \frac{i\xi_1 \xi_2 \log(\|\alpha\|/\|\beta\|)}{\Re e^2 G} & \frac{1}{\Re e G} \end{pmatrix}$$

is (well defined and) l.c.K on $(S^1 \times S^3, J_{\alpha,\beta})$.

Proof: For fixed h , the Cauchy problem

$$\begin{cases} \frac{l'^2 + l''}{l'} = h \\ l'(\theta_0) > 0 \end{cases} \quad (14)$$

satisfies the local existence theorem for any $\theta_0 \in \mathbb{R}$. This means we can find an open covering U of \mathbb{R} and functions $l_{\mathcal{U}}: \mathcal{U} \rightarrow \mathbb{R}$ which satisfy the equation. Moreover U and $\{l_{\mathcal{U}}\}_{\mathcal{U} \in U}$ can be chosen so that h is increasing for any $\mathcal{U} \in U$; finally, note that, since h is positive, so is $l'^2 + l''$, and this gives the required family. \blacksquare

The previous theorem extends the corollary 1 of [GO98].

The Lee form of the metric $g_{\alpha,\beta}^h$ associated to a function h is given by (see (1) and (13))

$$\omega = -d \log(2\Phi_{\alpha,\beta}\pi l') = -\frac{l'^2 + l''}{l'} e^1 = -h e^1.$$

Remark 3.4 If $h: S^1 \rightarrow \mathbb{R}^+$ is constant, a (global) solution of the Cauchy problem (14) is given by $l(\theta) = h\theta$, and the potential of the corresponding $g_{\alpha,\beta}^h$ is given by (see (12))

$e^{h\theta}$. In [GO98] the potential is $e^{l(\log \|\alpha\| + \log \|\beta\|)\theta/(2\pi)}$, where l is any positive real number (see [GO98, after remark 3]): thus, for h constant, the constant l of [GO98] is given by

$$l = \frac{2\pi h}{\log \|\alpha\| + \log \|\beta\|}.$$

Remark 3.5 If $\|\alpha\| = \|\beta\|$, we get $\Re e G = \log \|\alpha\|$, $\log(\|\alpha\|/\|\beta\|) = 0$ and

$$g_{\alpha,\beta}^h = \frac{1}{\log \|\alpha\|} \begin{pmatrix} \frac{\pi h}{\log \|\alpha\|} & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus in the case $\|\alpha\| = \|\beta\|$ the family given by the theorem 3.3 coincide up to a constant with the family given by (9), where $k = \pi h / \log \|\alpha\|$.

For a general $h^{(3)}$ the Lee vector field B of $g_{\alpha,\beta}^h$ is

$$B = -4\pi e_1 + 2\Im m Ge_2 + 2\Im m(\xi_1\xi_2) \arg(\alpha/\beta)e_3 - 2\Re e(\xi_1\xi_2) \arg(\alpha/\beta)e_4$$

and the “six terms formula” ([KN69, proposition 2.3]) gives

$$\begin{aligned} g_{\alpha,\beta}^h(\nabla_{e_1}(B), e_1) &= -\frac{h' \|G\|^2}{2\Re e^2 G}, & g_{\alpha,\beta}^h(\nabla_{e_2}(B), e_2) &= -\frac{2h'\pi^2}{\Re e^2 G}, \\ g_{\alpha,\beta}^h(\nabla_{e_1}(B), e_2) &= g_{\alpha,\beta}^h(\nabla_{e_2}(B), e_1) = -\frac{h'\Im m G\pi}{\Re e^2 G}, \\ g_{\alpha,\beta}^h(\nabla_{e_i}(B), e_j) &= 0 \quad \text{otherwise.} \end{aligned}$$

So the following holds:

Theorem 3.6 *The metric $g_{\alpha,\beta}^h$ of theorem 3.3 is Vaisman if and only if h is constant.*

4 Some foliations on $S^1 \times S^3$

On any l.c.K. manifold (M, J, g) with a never-vanishing Lee form ω , the following canonical distributions are given:

1. The kernel of the Lee form: since $d\omega = 0$, and

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]) \quad X, Y \in \mathfrak{X}(M)$$

such a distribution is integrable, so we get a codimension 1 foliation that we shall denote by \mathcal{F} ;

⁽³⁾If h is not constant the metric $g_{\alpha,\beta}^h$ restricted to the fibre S^3 of the projection $S^1 \times S^3 \rightarrow S^1$ does depend on θ , so the argument of [GO98, proposition 3 and corollary 2] doesn't apply.

2. the flow of the Lee vector field B , dual via g of ω : since

$$g(B, X) = \omega(X) = 0 \quad \text{for every } X \in \ker \omega$$

this foliation is in fact \mathcal{F}^\perp ;

3. the flow of the vector field JB : this foliation will be denoted by $J\mathcal{F}^\perp$;

4. the 2-dimensional distribution spanned by B and JB is $\mathcal{F}^\perp \oplus J\mathcal{F}^\perp$: whenever the Lee form is parallel, this distribution is integrable (see e.g. [CP85, theorem 4.3], but this condition is not necessary, as we shall see), and moreover, it defines a Riemannian foliation (see [DO98, Theorem 5.1]).

The notation is taken from [CP85] and [Pic90], where these and other related distributions are studied.

In our case, remark that since $\omega = he^1$ where h is strictly positive, ω is never-vanishing.

4.1 The foliation \mathcal{F}

The foliation \mathcal{F} is simply the S^3 spheres foliation given by the diffeomorphism $F_{\alpha,\beta}$: so in the parallel case -namely, for h constant- these S^3 are totally geodesic submanifolds of $(S^1 \times S^3, g_{\alpha,\beta}^h)$ (see [CP85, lemma 4.1]).

4.2 The foliations \mathcal{F}^\perp , $J\mathcal{F}^\perp$ and $\mathcal{F}^\perp \oplus J\mathcal{F}^\perp$

Let us consider the torus $S^1 \times S^1$ with coordinates (t_1, t_2) . The following is well known:

Lemma 4.1 *The curve in $S^1 \times S^1$ given by the linear functions*

$$t_1(t) = \gamma_1 + \delta_1 t \mod 2\pi, \quad t_2(t) = \gamma_2 + \delta_2 t \mod 2\pi \tag{15}$$

is

1. compact if $\delta_2/\delta_1 \in \mathbb{Q}$;

2. dense in $S^1 \times S^1$ otherwise.

In the case 1 of the previous lemma, the curve (15) is called a *toral knot of type δ_2/δ_1* (see figure 1).

Let us now fix a point (Θ, Ξ_1, Ξ_2) in $S^1 \times S^3$. To study the leaves passing through (Θ, Ξ_1, Ξ_2) of \mathcal{F}^\perp and $J\mathcal{F}^\perp$ in the case $\Xi_1 \Xi_2 \neq 0$, we define the submanifold T of S^3 as the product of two circles of radius respectively $\|\Xi_1\|$ e $\|\Xi_2\|$:

$$T \stackrel{\text{def}}{=} T(\Xi_1, \Xi_2) \stackrel{\text{def}}{=} S^1_{\|\Xi_1\|} \times S^1_{\|\Xi_2\|} \subset \mathbb{C} \times \mathbb{C}$$

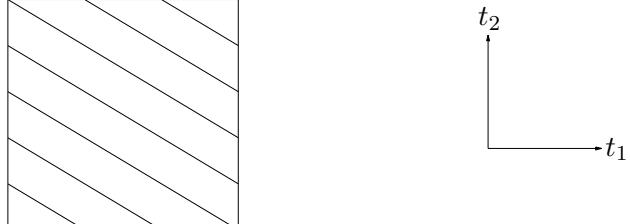


Figure 1: toral knot of type $-\frac{3}{5}$.

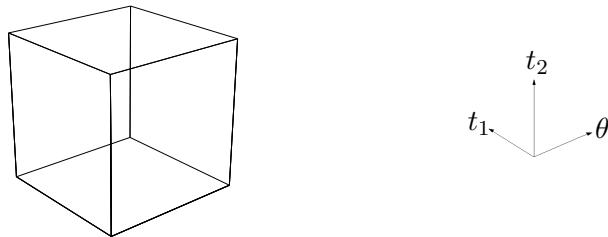


Figure 2: the 3-torus $S^1 \times T$.

and we denote by t_1 and t_2 the coordinates on the torus T given by

$$\xi_1(t_1) = \Xi_1 e^{it_1}, \quad \xi_2(t_2) = \Xi_2 e^{it_2}. \quad (16)$$

We then consider in $S^1 \times S^3$ the real 3-dimensional torus $S^1 \times T$, containing the point (Θ, Ξ_1, Ξ_2) ; a curve in this 3-torus is given by

$$\theta = \theta(t) \mod 2\pi, \quad t_1 = t_1(t) \mod 2\pi, \quad t_2 = t_2(t) \mod 2\pi.$$

We can visualize $S^1 \times T$ as a cube with identifications (see figure 2).

4.2.1 The foliation \mathcal{F}^\perp

The Lee vector field of $g_{\alpha,\beta}^h$ is

$$B = -4\pi e_1 + 2\operatorname{Im} Ge_2 + 2\operatorname{Im}(\xi_1 \xi_2) \arg(\alpha/\beta) e_3 - 2\operatorname{Re}(\xi_1 \xi_2) \arg(\alpha/\beta) e_4$$

-we remark that this vector field does not depend on h - and using (6) we get

$$B = -4\pi e_1 + 2i(\xi_1 \arg \alpha, \xi_2 \arg \beta).$$

By means of $F_{\alpha,\beta}$ (formula (7)) we can read the Lee vector field as a vector field in $\mathbb{C}^2 - 0$, where it becomes (see also [GO98, formula (23)])

$$B = -2(z_1 \log \|\alpha\|, z_2 \log \|\beta\|). \quad (17)$$

This last expression is easily integrable, and we obtain

$$(z_1(t), z_2(t)) = (z_1(0)e^{-2t \log \|\alpha\|}, z_2(0)e^{-2t \log \|\beta\|}) \quad t \in \mathbb{R}, \quad (18)$$

where the initial condition $(z_1(0), z_2(0))$ is tied to (Θ, Ξ_1, Ξ_2) by

$$\Xi_1 e^{\frac{\Theta \log \alpha}{2\pi}} = z_1(0), \quad \Xi_2 e^{\frac{\Theta \log \beta}{2\pi}} = z_2(0). \quad (19)$$

We can now pull the integral curve back to $S^1 \times S^3$ via $F_{\alpha, \beta}$: setting

$$\xi_1(t) e^{\frac{\theta(t) \log \alpha}{2\pi}} = z_1(0) e^{-2t \log \|\alpha\|}, \quad \xi_2(t) e^{\frac{\theta(t) \log \beta}{2\pi}} = z_2(0) e^{-2t \log \|\beta\|}, \quad (20)$$

we obtain the following equation for $\theta(t)$:

$$\|z_1(0)\|^2 e^{-\log \|\alpha\|(4t + \frac{\theta(t)}{\pi})} + \|z_2(0)\|^2 e^{-\log \|\beta\|(4t + \frac{\theta(t)}{\pi})} = 1;$$

calling $x = x(\Theta, \Xi_1, \Xi_2)$ the unique solution of the equation

$$\|z_1(0)\|^2 x^{\log \|\alpha\|} + \|z_2(0)\|^2 x^{\log \|\beta\|} = 1, \quad (21)$$

we obtain

$$\theta(t) = -\pi(\log x + 4t) \quad (22)$$

and together with (20) and (19) we get

$$\xi_1(t) = z_1(0) e^{\frac{\log x \log \alpha}{2}} e^{2it \arg \alpha} = \Xi_1 e^{2it \arg \alpha}, \quad \xi_2(t) = z_2(0) e^{\frac{\log x \log \beta}{2}} e^{2it \arg \beta} = \Xi_2 e^{2it \arg \beta}. \quad (23)$$

We distinguish two kinds of points in $S^1 \times S^3$. If $\Xi_1 \Xi_2 = 0$, say $\Xi_2 = 0$, the leaf given by (22) and (23) is contained in $S^1 \times \{(\xi_1, \xi_2) \in S^3 : \xi_2 = 0\}$. According to lemma 4.1, if $\arg \alpha$ is a rational multiple of π , the leaf is compact; otherwise it is dense in $S^1 \times \{(\xi_1, \xi_2) \in S^3 : \xi_2 = 0\}$. If $\Xi_1 \Xi_2 \neq 0$, from equations (23) we obtain that $\xi_1(t)$ and $\xi_2(t)$ have a constant positive length for every t , so the leaf is contained in the real 3-torus $S^1 \times T$ defined at page 13. Once observed that $\Theta = -\pi \log x \bmod 2\pi$, the equations (22) and (23) can be written as

$$\theta(t) = \Theta - 4\pi t \bmod 2\pi, \quad t_1(t) = 2t \arg \alpha \bmod 2\pi, \quad t_2(t) = 2t \arg \beta \bmod 2\pi. \quad (24)$$

In order to study the compactness of the leaves we remark that:

1. the leaf projected on T is given by

$$t_1(t) = 2t \arg \alpha \bmod 2\pi, \quad t_2(t) = 2t \arg \beta \bmod 2\pi, \quad (25)$$

and by lemma 4.1 this is a compact set if the ratio of $\arg \alpha$ to $\arg \beta$ is rational; otherwise it is dense in T . Since the projection from $S^1 \times T$ on T is a closed map, we can infer that if the ratio of $\arg \alpha$ to $\arg \beta$ is not rational then the leaf is not compact. If this ratio is rational, then the projected set is a toral knot of type $\arg \alpha / \arg \beta$ (see figure 3);

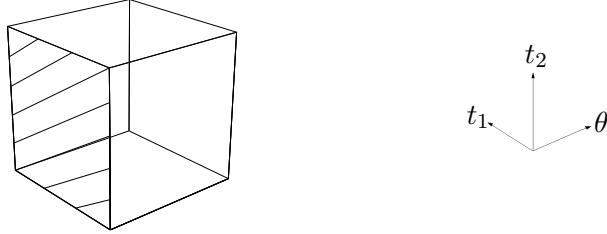


Figure 3: projection of the leaf of $J\mathcal{F}^\perp$ to T : case $\arg \alpha / \arg \beta \in \mathbb{Q}$.

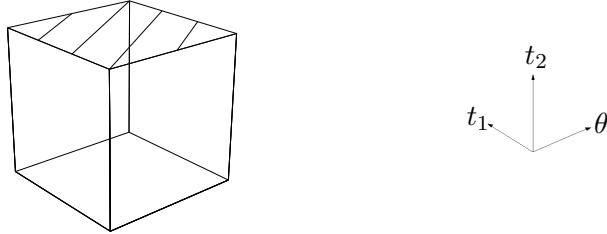


Figure 4: projection of the leaf of $J\mathcal{F}^\perp$ to $\{t_2 = 0\}$: case $(\arg \alpha) / \pi \in \mathbb{Q}$.

2. the projection of the leaf on the face $t_2 = 0$ of the cube in figure 2 is given by

$$\theta(t) = \Theta - 4\pi t \pmod{2\pi}, \quad t_1(t) = 2t \arg \alpha \pmod{2\pi},$$

and lemma 4.1 gives the condition $(\arg \alpha) / \pi \in \mathbb{Q}$ (see figure 4);

3. in the same way, if we consider the projection on the face $t_1 = 0$, we obtain $(\arg \beta) / \pi \in \mathbb{Q}$ (see figure 5).

We then have the three following *necessary* conditions for the compactness of the leaf:

$$\arg \alpha \in \mathbb{Q}\pi; \quad \arg \beta \in \mathbb{Q}\pi; \quad \arg \alpha / \arg \beta \in \mathbb{Q}, \quad (26)$$

where any two of them obviously imply the third. Let us show that the conditions (26) are also *sufficient* to obtain the compactness of the leaf. If the (26) hold, we can choose coprime integers l and k such that

$$\frac{\arg \alpha}{\arg \beta} = \frac{l}{k}.$$

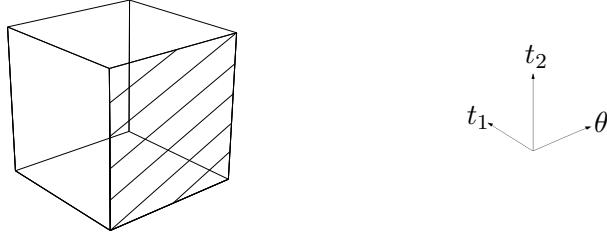


Figure 5: projection of the leaf of $J\mathcal{F}^\perp$ to $\{t_1 = 0\}$: case $(\arg \beta) / \pi \in \mathbb{Q}$.

The equations (25) define a closed curve with period $l\pi/\arg \alpha (=k\pi/\arg \beta)$, and the leaf is closed whenever the $\theta(t)$ given by equations (24) also has a period that is an integer multiple of $l\pi/\arg \alpha$. If we choose integers p and q such that $(\arg \alpha)/\pi = p/q$, it is straightforward to check that $pl\pi/\arg \alpha$ is a period of $\theta(t)$, and the proof is complete. To summarize:

Theorem 4.2 *Given the 1-dimensional foliation \mathcal{F}^\perp on $(S^1 \times S^3, J_{\alpha,\beta}, g_{\alpha,\beta}^h)$ the following holds:*

1. *for every α and β the leaf through the point $(\Theta, \Xi_1, 0)$ (respectively $(\Theta, 0, \Xi_2)$) is a subset of $S^1 \times \{(\xi_1, \xi_2) \in S^3 : \xi_2 = 0\}$ (respectively $S^1 \times \{(\xi_1, \xi_2) \in S^3 : \xi_1 = 0\}$). This leaf is*

- *compact if $\arg \alpha \in \mathbb{Q}\pi$ (respectively $\arg \beta \in \mathbb{Q}\pi$);*
- *dense in $S^1 \times \{(\xi_1, \xi_2) \in S^3 : \xi_2 = 0\}$ (respectively in $S^1 \times \{(\xi_1, \xi_2) \in S^3 : \xi_1 = 0\}$) otherwise;*

2. *for every α and β the leaf through the point (Θ, Ξ_1, Ξ_2) , where $\Xi_1 \Xi_2 \neq 0$, is a subset of $S^1 \times T$, where T is the torus in the factor S^3 of $S^1 \times S^3$ given by (16). This leaf is*

- *compact if any two of the (26) hold;*
- *non compact otherwise;*

if the leaf is not compact, then its projection on T is

- *a toral knot of type $\arg \alpha/\arg \beta$ if this ratio is rational;*
- *dense in T otherwise.*

4.2.2 The foliation $J\mathcal{F}^\perp$

The anti Lee vector field JB is given by

$$JB = -2\Re e Ge_2 - 2\Im m(\xi_1 \xi_2) \log \|\alpha/\beta\| e_3 + 2\Re e(\xi_1 \xi_2) \log \|\alpha/\beta\| e_4$$

-it is independent of h - and again by (6) and (7) we get

$$JB = -2i(\xi_1 \log \|\alpha\|, \xi_2 \log \|\beta\|) = -2i(z_1 \log \|\alpha\|, z_2 \log \|\beta\|),$$

so the integral curves are

$$(z_1(s), z_2(s)) = (z_1(0)e^{-2is \log \|\alpha\|}, z_2(0)e^{-2is \log \|\beta\|}).$$

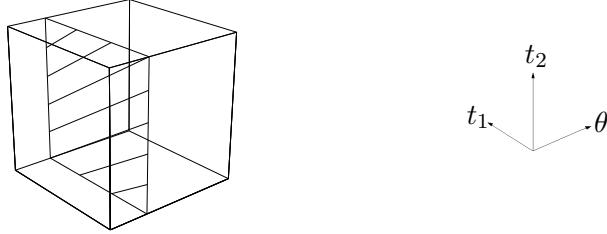


Figure 6: the leaf of $J\mathcal{F}^\perp$, case $\log \|\alpha\|/\log \|\beta\| \in \mathbb{Q}$.

These formulas are profoundly different from the previous ones, because of the complex exponent: in fact we have

$$\theta(s) = -\pi \log x,$$

where x is a solution of (21), and

$$\begin{aligned}\xi_1(s) &= z_1(0)e^{\frac{\log x \log \alpha}{2}} e^{-2is \log \|\alpha\|} = \Xi_1 e^{-2is \log \|\alpha\|}, \\ \xi_2(s) &= z_2(0)e^{\frac{\log x \log \beta}{2}} e^{-2is \log \|\beta\|} = \Xi_2 e^{-2is \log \|\beta\|}.\end{aligned}$$

If $\Xi_1 \Xi_2 = 0$, say $\Xi_2 = 0$, the leaf through $(\Theta, \Xi_1, 0)$ is $\{\Theta\} \times \{(\xi_1, \xi_2) \in S^3 : \xi_2 = 0\}$, so it is closed. If $\Xi_1 \Xi_2 \neq 0$, we again deduce that $\xi_1(s)$ and $\xi_2(s)$ have constant positive length, so the leaf through (Θ, Ξ_1, Ξ_2) is a subset of $\{\Theta\} \times T$, where T is still given by (16) (see figure 6).

We have thus obtained

Theorem 4.3 *Given the 1-dimensional foliation $J\mathcal{F}^\perp$ on $(S^1 \times S^3, J_{\alpha, \beta}, g_{\alpha, \beta}^h)$, the following holds:*

1. *for every α and β the leaf through the point $(\Theta, \Xi_1, 0)$ (respectively $(\Theta, 0, \Xi_2)$) is $\{\Theta\} \times \{(\xi_1, \xi_2) \in S^3 : \xi_2 = 0\}$ (respectively $\{\Theta\} \times \{(\xi_1, \xi_2) \in S^3 : \xi_1 = 0\}$), so it is compact;*
2. *for every α and β the leaf through the point (Θ, Ξ_1, Ξ_2) , where $\Xi_1 \Xi_2 \neq 0$, is a subset of $\{\Theta\} \times T$, where T is the torus in the factor S^3 of $S^1 \times S^3$ given by (16). This leaf is
 - a toral knot of type $\log \|\alpha\|/\log \|\beta\|$ if this ratio is rational;
 - dense in $\{\Theta\} \times T$ otherwise.*

4.2.3 The foliation $\mathcal{F}^\perp \oplus J\mathcal{F}^\perp$

The most interesting distribution is the one generated by both the Lee and the anti Lee vector fields: these planes are clearly closed with respect to J , so if the distribution is integrable the integral surfaces are complex curves.

Theorem 4.4 *The distribution $\mathcal{F}^\perp \oplus J\mathcal{F}^\perp$ is integrable. Moreover this distribution only depends on α and β .*

Proof: It is well known (see [CP85]) that if the Lee form is parallel then the distribution is integrable: now for the $g_{\alpha,\beta}^h$ we recall that

$$B = -2(z_1 \log \|\alpha\|, z_2 \log \|\beta\|), \quad JB = -2i(z_1 \log \|\alpha\|, z_2 \log \|\beta\|),$$

and these expressions are clearly independent of the function h , so for a fixed α and β we get a unique distribution on $S^1 \times S^3$ this coincides with the one induced by the Vaisman metric given by constant h , and is thus integrable. ■

Definition 4.5 We call $\mathcal{E}_{\alpha,\beta}$ the unique foliation given by theorem 4.4.

The following theorem gives an explicit description of the leaves of $\mathcal{E}_{\alpha,\beta}$:

Theorem 4.6 *The foliation $\mathcal{E}_{\alpha,\beta}$ on $S^1 \times S^3$ is described by the following properties:*

1. *for every α and β the leaf through the point $(\Theta, \Xi_1, 0)$ (respectively $(\Theta, 0, \Xi_2)$) is $S^1 \times \{(\xi_1, \xi_2) \in S^3 : \xi_2 = 0\}$ (respectively $S^1 \times \{(\xi_1, \xi_2) \in S^3 : \xi_1 = 0\}$), and it is thus compact;*
2. *for every α and β the leaf through the point (Θ, Ξ_1, Ξ_2) , where $\Xi_1 \Xi_2 \neq 0$, is a subset of $S^1 \times T$, where T is the torus in the factor S^3 of $S^1 \times S^3$ given by (16). This leaf is*
 - *compact if there exist integers m and n such that $\alpha^m = \beta^n$: in this case the leaf is a Riemann surface of genus one \mathbb{C}/Λ , where Λ is the lattice in \mathbb{C} generated by the vectors v and w given by (31);*
 - *non compact otherwise, and in this case it is dense in $S^1 \times T$.*

Proof: We consider the 2-dimensional real distribution as a field of 1-dimensional complex lines generated by B . We remark that if in the expression (18) we formally substitute with the real parameter t a complex parameter w we obtain

$$(z_1(w), z_2(w)) = (z_1(0)e^{-2w \log \|\alpha\|}, z_2(0)e^{-2w \log \|\beta\|}), \quad (27)$$

which results in a complex parametrization of the integral surface of $\mathcal{E}_{\alpha,\beta}$ passing through (Θ, Ξ_1, Ξ_2) , in the coordinates $[z_1, z_2]$. Again, as in the proof of theorem 4.2, we find a parametrization of the same leaf in the coordinates (θ, ξ_1, ξ_2) :

$$\begin{aligned} \theta(w) &= \Theta - 4\pi \operatorname{Re} w \mod 2\pi, \\ \xi_1(w) &= \Xi_1 e^{2i \arg \alpha \operatorname{Re} w} e^{-2i \log \|\alpha\| \operatorname{Im} w}, \\ \xi_2(w) &= \Xi_2 e^{2i \arg \beta \operatorname{Re} w} e^{-2i \log \|\beta\| \operatorname{Im} w}. \end{aligned} \quad (28)$$

The simplest case $\Xi_1\Xi_2 = 0$ follows from the equations (28). So we can suppose $\Xi_1\Xi_2 \neq 0$, and in this case the leaf is a subset of $S^1 \times T$, where T is given by (16). Setting $(t, s) \stackrel{\text{def}}{=} (\Re w, \Im w)$, the equations (28) become

$$\begin{aligned}\theta(t, s) &= \Theta - 4\pi t \pmod{2\pi}, \\ t_1(t, s) &= 2(\arg \alpha t - \log \|\alpha\|s) \pmod{2\pi}, \\ t_2(t, s) &= 2(\arg \beta t - \log \|\beta\|s) \pmod{2\pi}.\end{aligned}\tag{29}$$

Call N this leaf, and consider $N \cap (\{\Theta\} \times T)$. We observe that $\theta(t) = \Theta$ is equivalent to $t = m/2$ where m is an integer: call N_m the curve given by the equations

$$\begin{aligned}\theta\left(\frac{m}{2}, s\right) &= \Theta \pmod{2\pi}, \\ t_1\left(\frac{m}{2}, s\right) &= 2\left(\arg \alpha \frac{m}{2} - \log \|\alpha\|s\right) \pmod{2\pi}, \\ t_2\left(\frac{m}{2}, s\right) &= 2\left(\arg \beta \frac{m}{2} - \log \|\beta\|s\right) \pmod{2\pi}.\end{aligned}$$

Clearly $N \cap (\{\Theta\} \times T)$ is the union of the curves N_m for $m \in \mathbb{Z}$. By lemma 4.1 we know that N_m is dense in $\{\Theta\} \times T$ whenever $\log \|\alpha\|/\log \|\beta\|$ is irrational: $N \cap (\{\Theta\} \times T)$ is then *a fortiori* dense in $\{\Theta\} \times T$, and it is not $\{\Theta\} \times T$ since it does not contain for instance the points

$$\begin{aligned}\theta &= \Theta \pmod{2\pi}, \\ t_1(s) &= 2\left(\arg \alpha \frac{2m+1}{4} - \log \|\alpha\|s\right) \pmod{2\pi}, \\ t_2(s) &= 2\left(\arg \beta \frac{2m+1}{4} - \log \|\beta\|s\right) \pmod{2\pi}.\end{aligned}$$

We can use this argument for all θ , so in this case N is dense in $S^1 \times T$. Otherwise if $\log \|\alpha\|/\log \|\beta\|$ is rational, the intersection of N with $\{\theta\} \times T$ is the union of toral knots of type $\log \|\alpha\|/\log \|\beta\|$.

Let us now consider the intersection of N with the surface given by $t_2 = 0$: after observing that $t_2 = 0$ is equivalent to $s = (t \arg \beta - m\pi)/\log \|\beta\|$ for m integer, let us call N_m the curve given by

$$\begin{aligned}\theta\left(t, \frac{t \arg \beta - m\pi}{\log \|\beta\|}\right) &= -\pi \log x - 4\pi t \pmod{2\pi}, \\ t_1\left(t, \frac{t \arg \beta - m\pi}{\log \|\beta\|}\right) &= 2\left(\arg \alpha t - \log \|\alpha\| \frac{t \arg \beta - m\pi}{\log \|\beta\|}\right) \pmod{2\pi}, \\ t_2\left(t, \frac{t \arg \beta - m\pi}{\log \|\beta\|}\right) &= 0 \pmod{2\pi},\end{aligned}$$

(see figure 4). In this case lemma 4.1 shows that every N_m is dense in $S^1 \times \{(t_1, 0) \in T\}$ whenever $(\arg \alpha - \arg \beta \log \|\alpha\|/\log \|\beta\|)/\pi$ is irrational: the same argument for $t_2 \neq 0$ shows that in this case N is dense in $S^1 \times T$.

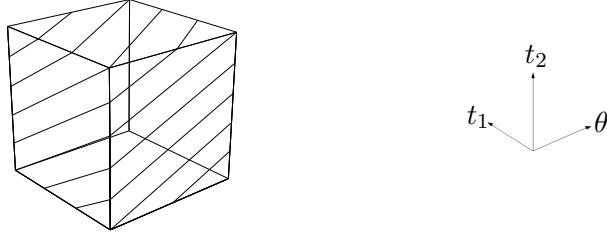


Figure 7: intersection of the leaf with the faces of $S^1 \times T$: case $\arg \alpha - \arg \beta \log \|\alpha\| / \log \|\beta\| \in \mathbb{Q}\pi$ and $\log \|\alpha\| / \log \|\beta\| \in \mathbb{Q}$.

We are then left to the case

$$\frac{\arg \alpha - \arg \beta \log \|\alpha\| / \log \|\beta\|}{\pi} \in \mathbb{Q}, \quad \frac{\log \|\alpha\|}{\log \|\beta\|} \in \mathbb{Q}$$

namely

$$\frac{k \arg \alpha - l \arg \beta}{\pi} = \frac{p}{q}, \quad \frac{\log \|\alpha\|}{\log \|\beta\|} = \frac{l}{k} \quad (30)$$

where l, k, p and q are integers and $(p, q) = (l, k) = 1$: in this case the intersection of N with the faces of the figure 2 is a union of closed curves (see figure 7).

Choose two integers b and c such that $bk - cl = 1$. Set

$$q' \stackrel{\text{def}}{=} \begin{cases} q & \text{if } p \text{ is odd} \\ q/2 & \text{if } p \text{ is even} \end{cases}, \quad p' \stackrel{\text{def}}{=} \begin{cases} p & \text{if } p \text{ is odd} \\ p/2 & \text{if } p \text{ is even} \end{cases}$$

and remark that in this case the map

$$\begin{aligned} F : \quad \mathbb{R}^2 &\longrightarrow N \subset S^1 \times T \\ (t, s) &\longmapsto (\theta(t, s), t_1(t, s), t_2(t, s)) \end{aligned}$$

is invariant with respect to the action on \mathbb{R}^2 of the lattice $\Lambda \stackrel{\text{def}}{=} v\mathbb{Z} \oplus w\mathbb{Z}$ (see figure 8) where

$$v = (q', \frac{q' \arg \beta - p' c \pi}{\log \|\beta\|}), \quad w = (0, \frac{k \pi}{\log \|\beta\|}). \quad (31)$$

So we may consider the diagram

$$\begin{array}{ccc} \mathbb{C} & & \\ p \downarrow & \searrow F & \\ \mathbb{C}/\Lambda & \xrightarrow{\bar{F}} & N \end{array} \quad (32)$$

where p is the canonical projection of \mathbb{C} on \mathbb{C}/Λ and \bar{F} is the quotient map of F . Obviously \bar{F} is onto, and the leaf $N = \bar{F}(\mathbb{C}/\Lambda)$ is compact. Moreover, since $F' = B \neq 0$, \bar{F} is a

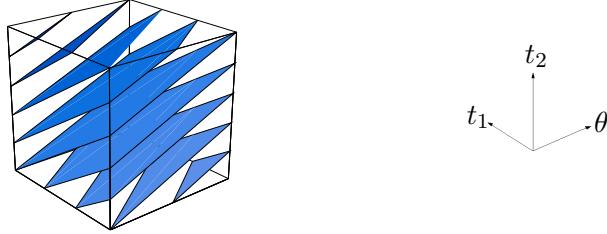


Figure 8: the compact leaf in the case $\arg \alpha - \arg \beta \log \|\alpha\| / \log \|\beta\| \in \mathbb{Q}\pi$ and $\log \|\alpha\| / \log \|\beta\| \in \mathbb{Q}$.

local diffeomorphism; this implies that N , being the image of a compact manifold via a local diffeomorphism, is a submanifold of $H_{\alpha,\beta}$. Thus N , being closed with respect to $J_{\alpha,\beta}$, is a compact Riemann surface and its genus is one, since it supports a non-vanishing vector field. Furthermore \bar{F} is holomorphic, because, with the chosen parametrization, the horizontal and the vertical axes of \mathbb{C} are just the integral curves respectively of B and JB . It follows that \bar{F} is a non ramified covering. But it is straightforward to check that \bar{F} is injective also, so it is a biholomorphism.

Lemma 4.7 shows that the conditions (30) coincide with the condition $\alpha^m = \beta^n$ and the theorem is proved. \blacksquare

Lemma 4.7 *The conditions (30) are equivalent to the existence of integers m and n , where $m/n = k/l$, such that $\alpha^m = \beta^n$.*

Proof: The existence of integers m and n such that $m/n = k/l$ and $\alpha^m = \beta^n$ is equivalent to

$$\frac{\log \|\alpha\|}{\log \|\beta\|} = \frac{n}{m} = \frac{l}{k} \quad \text{and} \quad \{m \arg \alpha + 2r\pi\}_{r \in \mathbb{Z}} = \{n \arg \beta + 2s\pi\}_{s \in \mathbb{Z}}. \quad (33)$$

These conditions obviously imply (30).

Vice versa, from (30) we obtain

$$2qk \arg \alpha + 2r\pi = 2ql \arg \beta + 2\pi(p+r) \quad \text{for every integer } r;$$

so, setting $m \stackrel{\text{def}}{=} 2qk$ and $n \stackrel{\text{def}}{=} 2ql$, we get (33). \blacksquare

The proof of theorem 4.6 allows us to complete the description of the foliation when the leaves are not compact:

Corollary 4.8 *When α and β do not satisfy (30), the saturated components of $\mathcal{E}_{\alpha,\beta}$ are of two kinds:*

1. $S^1 \times \{(\xi_1, \xi_2) \in S^3 : \xi_2 = 0\}$ and $S^1 \times \{(\xi_1, \xi_2) \in S^3 : \xi_1 = 0\}$;
2. $S^1 \times T(\xi_1, \xi_2)$.

Remark 4.9 Because of (17), $\mathcal{E}_{\alpha,\beta}$ is linear in the classification recently given by D. Mall in [Mal98].

5 Elliptic fibrations on $S^1 \times S^3$

By the definition of Kodaira in [Kod64, 2], an *elliptic surface* is a complex fibre space of elliptic curves over a non singular algebraic curve, namely a map $\Xi: S \rightarrow \Delta$ where S is a complex surface, Δ is a non singular algebraic curve, Ψ is a holomorphic map and the generic fibre is a torus. The curve Δ is called the *base space* of S .

In theorem 4.6 we showed that, if $\alpha^m = \beta^n$ for some integers m and n , then $S^1 \times S^3$ is a fibre space of elliptic curves over a topological space Δ -the leaf space. Now we show that such a Δ is a non singular algebraic curve (actually $\mathbb{P}^1\mathbb{C}$) and that the projection Ψ is holomorphic with respect to this complex structure.

Theorem 5.1 *If $\alpha^m = \beta^n$ for some integers m and n , the leaf space Δ of the foliation in tori given on $S^1 \times S^3$ by the theorem 4.6 is homeomorphic to $\mathbb{P}^1\mathbb{C}$, and the projection $\Psi: S^1 \times S^3 \rightarrow \Delta$ is holomorphic with respect to the induced complex structure.*

Proof: By lemma 4.7 the hypothesis is equivalent to the conditions (30). Choose then the integers m and n minimal with respect to the property $\alpha^m = \beta^n$, and observe that this implies $m \arg \alpha = n \arg \beta + 2\pi c$, where c is an integer such that $\text{MCD}(m, n, c) = 1$, and consider the following map:

$$\begin{aligned} \tilde{h}: \quad S^1 \times S^3 &\longrightarrow \mathbb{P}^1\mathbb{C} \\ (\theta, \xi_1, \xi_2) &\longmapsto [e^{\theta i c} \xi_1^m : \xi_2^n]. \end{aligned}$$

It is an easy matter to verify that on $H_{\alpha,\beta}$ this map is nothing but the quotient of $\phi(z_1, z_2) \stackrel{\text{def}}{=} [z_1^m : z_2^n]$, and we obtain the diagram

$$\begin{array}{ccccc} & & \mathbb{C}^2 - 0 & & (34) \\ & \searrow & & \swarrow & \\ H_{\alpha,\beta} & \xrightarrow{F_{\alpha,\beta}^{-1}} & S^1 \times S^3 & \xrightarrow{\phi} & \mathbb{P}^1\mathbb{C} \\ & \downarrow \Psi & \downarrow \tilde{h} & \downarrow h & \\ \Delta & \xrightarrow{\dots} & \mathbb{P}^1\mathbb{C} & \xrightarrow{\dots} & \end{array}$$

We show that \tilde{h} is well defined on the leaf space, and that its quotient h is in fact the homeomorphism we are looking for:

1. h is well defined: if (θ, ξ_1, ξ_2) is on the leaf passing through (Θ, Ξ_1, Ξ_2) , then θ, ξ_1

and ξ_2 are of the form (see (28))

$$\begin{aligned}\theta(t, s) &= \Theta - 4\pi t \mod 2\pi, \\ \xi_1(t, s) &= \Xi_1 e^{2i \arg \alpha t} e^{-2i \log \|\alpha\| s}, \\ \xi_2(t, s) &= \Xi_2 e^{2i \arg \beta t} e^{-2i \log \|\beta\| s},\end{aligned}$$

and we get

$$(\theta(t, s), \xi_1(t, s), \xi_2(t, s)) \mapsto [e^{i(\Theta-4\pi t)c} \Xi_1^m e^{2itm \arg \alpha} : \Xi_2^n e^{2itn \arg \beta}]$$

that is

$$(\theta(t, s), \xi_1(t, s), \xi_2(t, s)) \mapsto [e^{i(\Theta-4\pi t)c+2it(m \arg \alpha - n \arg \beta)} \Xi_1^m : \Xi_2^n] = [e^{i\Theta c} \Xi_1^m : \Xi_2^n],$$

and the last member does not depend on t and s . Namely, \tilde{h} is constant on every leaf and h is well defined on Δ ;

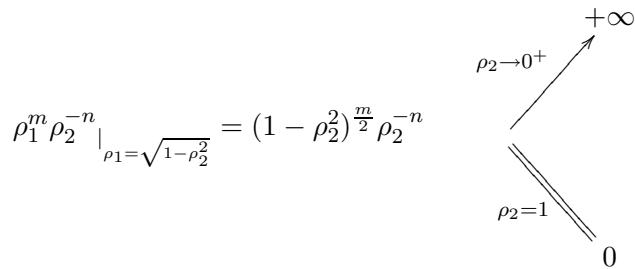
2. h is onto: $(\theta, 1, 0) \mapsto [1 : 0]$ and if we put $h(\theta, \xi_1, \xi_2) = [z_1 : z_2]$ where $z_2 \neq 0$ we obtain

$$z_1 z_2^{-1} = e^{i\theta c} \xi_1^m \xi_2^{-n}.$$

Using polar coordinates, that is, choosing real numbers ρ_1, ρ_2, θ_1 and θ_2 such that $\xi_1 = \rho_1 e^{i\theta_1}$ and $\xi_2 = \rho_2 e^{i\theta_2}$, the last member becomes

$$e^{i\theta c + m\theta_1 - n\theta_2} \rho_1^m \rho_2^{-n} \quad \text{where} \quad \rho_1^2 + \rho_2^2 = 1.$$

The exponent $\theta c + m\theta_1 - n\theta_2$ covers all the real numbers, and the map



covers all the positive real numbers, so \tilde{h} -and, consequently, h - is onto;

3. h is injective: suppose that $h(\theta, \xi_1, \xi_2) = h(\Theta, \Xi_1, \Xi_2)$ for two points (θ, ξ_1, ξ_2) and (Θ, Ξ_1, Ξ_2) on $S^1 \times S^3$. If $\xi_1 \Xi_1 = 0$, then ξ_1 and Ξ_1 must both of them be zero, whence (θ, ξ_1, ξ_2) and (Θ, Ξ_1, Ξ_2) lie on the same leaf. If $\xi_1 \Xi_1 \neq 0$, we can write

$$\frac{\xi_2^n}{e^{i\theta c} \xi_1^m} = \frac{\Xi_2^n}{e^{i\Theta c} \Xi_1^m}. \tag{35}$$

Let $\xi_1 = \rho_1 e^{i\eta_1}$, $\xi_2 = \rho_2 e^{i\eta_2}$, $\Xi_1 = P_1 e^{iH_1}$ and $\Xi_2 = P_2 e^{iH_2}$; the equation (35) becomes

$$\frac{\rho_2^n e^{i\eta_2 n}}{\rho_1^m e^{i(\theta c + \eta_1 m)}} = \frac{P_2^n e^{iH_2 n}}{P_1^m e^{i(\theta c + H_1 m)}},$$

that is

$$\begin{cases} \frac{\rho_2^n}{\rho_1^m} = \frac{P_2^n}{P_1^m}, \\ (\theta - \Theta)c + m(\eta_1 - H_1) - n(\eta_2 - H_2) = 0 \pmod{2\pi}. \end{cases} \quad (36)$$

The first equation in (36), together with $\rho_1^2 + \rho_2^2 = 1 = P_1^2 + P_2^2$, easily gives

$$\rho_1 = P_1 \quad \text{and} \quad \rho_2 = P_2. \quad (37)$$

In order to show that (θ, ξ_1, ξ_2) and (Θ, Ξ_1, Ξ_2) lie on the same leaf, we want to find two real numbers t and s such that

$$\begin{aligned} \theta &= \Theta - 4\pi t \pmod{2\pi}, \\ \xi_1 &= \Xi_1 e^{2(\arg \alpha t - \log \|\alpha\| s)}, \\ \xi_2 &= \Xi_2 e^{2(\arg \beta t - \log \|\beta\| s)}, \end{aligned} \quad (38)$$

that is, by using (37), we want to find two real numbers t and s satisfying

$$\begin{cases} 4\pi t = \Theta - \theta \pmod{2\pi}, \\ 2 \arg \alpha t - 2 \log \|\alpha\| s = \eta_1 - H_1 \pmod{2\pi}, \\ 2 \arg \beta t - 2 \log \|\beta\| s = \eta_2 - H_2 \pmod{2\pi}. \end{cases}$$

The determinant of

$$\begin{pmatrix} 4\pi & 0 & \Theta - \theta \\ 2 \arg \alpha & -2 \log \|\alpha\| & \eta_1 - H_1 \\ 2 \arg \beta & -2 \log \|\beta\| & \eta_2 - H_2 \end{pmatrix}$$

is zero, because the second equation of (36) gives us that

$$m(\text{second row}) - n(\text{third row}) = c(\text{first row}),$$

and the injectivity of h is proved.

From 1, 2 and 3 we obtain that $h: \Delta \rightarrow \mathbb{P}^1 \mathbb{C}$ is a bijective continuous map, and so is a homeomorphism because of the compactness of Δ . At least, Ψ is holomorphic with respect to the induced complex structure -that is, \tilde{h} is holomorphic- because the map ϕ in the diagram (34) is holomorphic. \blacksquare

6 Regularity of $\mathcal{E}_{\alpha,\beta}$ and orbifold structure on Δ

A *quasi-regular foliation* is a foliation \mathcal{F} on a smooth manifold M such that for each point p of M there is a natural number $N(p)$ and a Frobenius chart U (namely, a \mathcal{F} -flat cubical neighborhood) where each leaf of \mathcal{F} intersects U in $N(p)$ slices, if any. If $N(p) = 1$ for all p , then \mathcal{F} is called a *regular foliation* (see for instance [BG98]). For a compact manifold M , the assumption that the foliation is quasi-regular is equivalent to the assumption that all leaves are compact. A Riemannian foliation with compact leaves induces a natural orbifold structure on the leaf space (see [Mol88, Proposition 3.7]). This is the case we are concerned with, since by [DO98, Theorem 5.1] $\mathcal{E}_{\alpha,\beta}$ is Riemannian.

Theorem 6.1 *The foliation $\mathcal{E}_{\alpha,\beta}$ is quasi-regular if and only if $\alpha^m = \beta^n$ for some integers m and n ; in this case $N(\Theta, \Xi_1, \Xi_2) = 1$ if $\Xi_1 \Xi_2 \neq 0$, whereas $N(\Theta, 0, \Xi_2) = m$ and $N(\Theta, \Xi_1, 0) = n$. In particular, the foliation $\mathcal{E}_{\alpha,\beta}$ is regular if and only if $\alpha = \beta$.*

Proof: By theorem 4.6 we know that all leaves are compact if and only if $\alpha^m = \beta^n$, and for the points (Θ, Ξ_1, Ξ_2) where $\Xi_1 \Xi_2 \neq 0$ the thesis is given by the figure 8. We are then left to the points $(\Theta, 0, \Xi_2)$ and $(\Theta, \Xi_1, 0)$, when $\alpha^m = \beta^n$. We look at the points $(\Theta, \Xi_1, 0)$, the study of the other ones being analogous.

We remark that the figure 8 is 3-dimensional, and in order to visualize the 4-dimensional neighborhood of a point of $S^1 \times S^3$ we need another 3-dimensional description of the foliation $\mathcal{E}_{\alpha,\beta}$: consider the stereographic projection

$$\begin{aligned}\phi : \quad S^3 - (0, 0, 0, 1) &\longrightarrow \mathbb{R}^3 \\ (x_1, x_2, x_3, x_4) &\longmapsto \frac{1}{1-x_4}(x_1, x_2, x_3).\end{aligned}$$

It is easy to check that $\phi(T(\xi_1, \xi_2))$ is generated by the revolution around the y_3 -axis of the circle $C(\xi_1, \xi_2)$ in the y_2y_3 -plane centered in $(1/\|\xi_1\|, 0)$ with radius $\|\xi_2\|/\|\xi_1\|$. We are thus led to the figure 9.

By refining the computation in the proof of theorem 4.6, we see that any leaf intersects $T(\xi_1, \xi_2)$ along r toral knots of type l/k , r being the greatest common divisor of m and n . This means that each leaf contained in $T(\xi_1, \xi_2)$ intersects $C(\xi_1, \xi_2)$ in exactly $n = rl$ points. Now let

$$D_\rho \stackrel{\text{def}}{=} \bigcup_{\|\xi_2\|/\|\xi_1\| < \rho} C(\xi_1, \xi_2)$$

and let $U_{\delta,\rho}$ the piece of solid torus given by the revolution of angle $(-\delta, \delta)$ of D_ρ . The neighborhoods of $(\Theta, \Xi_1, 0)$ of the form $(\Theta - \varepsilon, \Theta + \varepsilon) \times U_{\delta,\rho}$ contain each leaf in $n = rl$ distinct connected components, and this ends the proof. \blacksquare

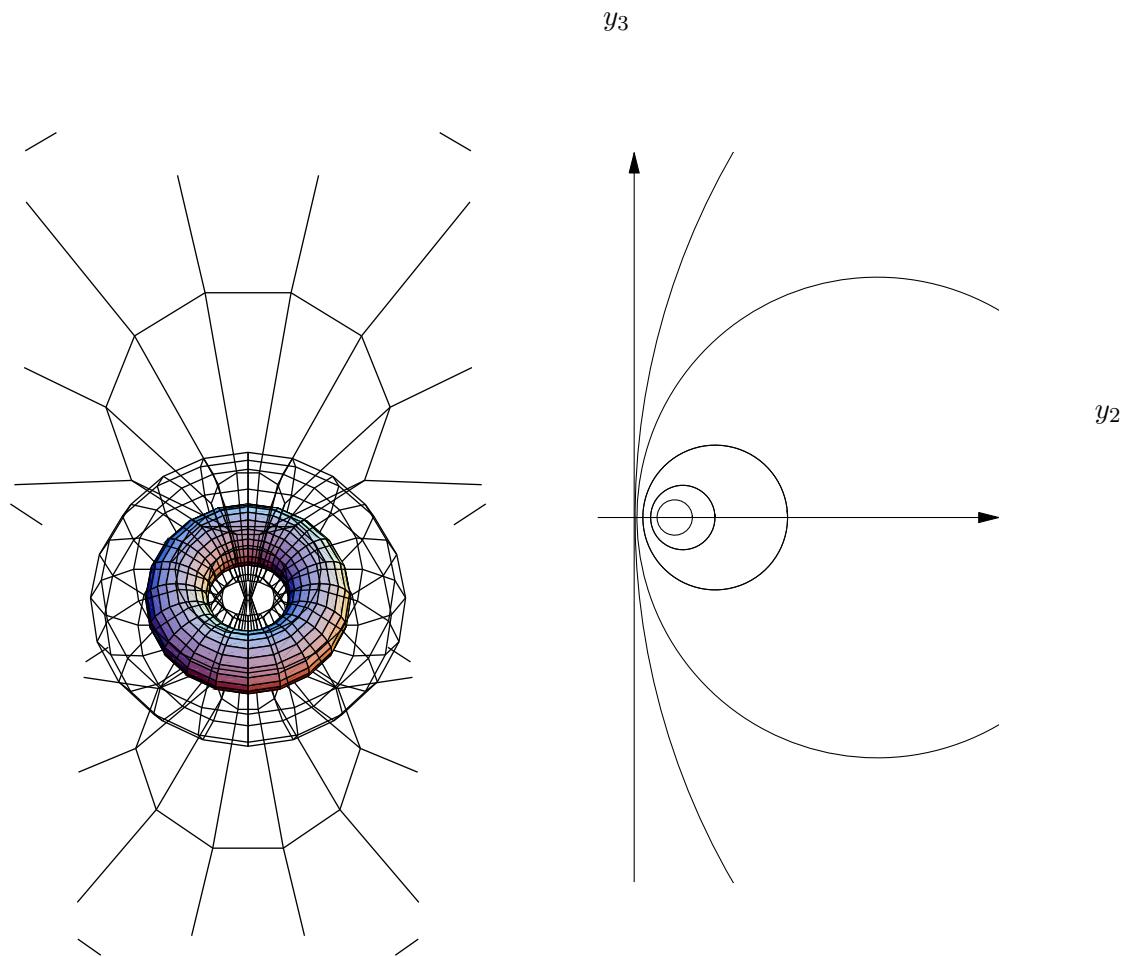


Figure 9: On the left, the partition of \mathbb{R}^3 in tori $T(\xi_1, \xi_2)$; on the right, the circles that generate the tori.

Remark 6.2 We thus have an orbifold structure on the leaf space Δ , with two conical points of order m and n , respectively (see [Mol88, Proposition 3.7]). In particular, a local chart around the leaf through $(\Theta, \Xi_1, 0)$ is given by D_ρ/Γ_n , Γ_n being the finite group generated by the rotation of angle $2\pi/n$.

Remark 6.3 In the preceding section we gave Δ a structure of complex curve; this does not contradict the orbifold structure, it simply means that the two structures are not isomorphic in the orbifold category. In fact, any 2-dimensional orbifold with only conical points is homeomorphic to a manifold.

Acknowledgements: The author wishes to thank Florin Belgun, Paul Gauduchon, Rosa Gini, Liviu Ornea, Marco Romito and Izu Vaisman for the useful conversations, and Andrew Swann for the clear explanation of a part of his paper [PPS93].

This paper is part of a Ph. D. thesis, and the author wishes in a special way to thank Paolo Piccinni for the motivation and the constant help.

References

- [Bel99] F. A. Belgun. *Géométrie conforme et géométrie CR en dimensions 3 et 4*. PhD thesis, Centre de Mathématiques, UMR 7640 CNRS, Ecole Polytechnique, 91128 Palaiseau cedex, France, 1999.
- [Bes81] A. Besse. *Géométrie Riemannienne en dimension 4*. Cedic, Paris, 1981.
- [BG98] C. P. Boyer and K. Galicki. On Sasakian-Einstein geometry. electronic preprint, November 1998.
- [BPV84] W. Barth, C. Peters, and A. Van de Ven. *Compact complex surfaces*. Springer-Verlag, Berlin Heidelberg New York Tokyo, 1984.
- [CP85] B. Y. Chen and P. Piccinni. The canonical foliations of a locally conformal Kähler manifold. *Ann. di Mat. Pura e Appl.*, 141:289–305, 1985.
- [DO98] S. Dragomir and L. Ornea. *Locally Conformal Kähler Geometry*, volume 155 of *Progress in Math.* Birkhäuser, Basel, 1998.
- [Gau81] P. Gauduchon. Surfaces de Hopf. Variétés presque complexes de dimension 4. In *Géométrie Riemannienne en dimension 4* [Bes81], pages 134–155.
- [GO98] P. Gauduchon and L. Ornea. Locally conformally Kähler metrics on Hopf surfaces. *Ann. Inst. Fourier*, 48:1107–1127, 1998.

[HL83] R. Harvey and H. Blaine Lawson, Jr. An intrinsic characterization of Kähler manifolds. *Invent. Math.*, 74:169–198, 1983.

[Kat75] M. Kato. Topology of Hopf surfaces. *J. Math. Soc. Japan*, 27(2):223–238, 1975.

[KN69] S. Kobayashi and K. Nomizu. *Foundations of differential geometry*, volume I and II. Interscience Publishers, New York, 1963 and 1969.

[Kod64] K. Kodaira. On the structure of compact complex analytic surfaces, I. *American J. Math.*, 86:751–798, 1964.

[Kod66] K. Kodaira. On the structure of compact complex analytic surfaces, II. *American J. Math.*, 88:682–721, 1966.

[Kod68a] K. Kodaira. On the structure of compact complex analytic surfaces, III. *American J. Math.*, 90:55–83, 1968.

[Kod68b] K. Kodaira. On the structure of compact complex analytic surfaces, IV. *American J. Math.*, 90:1048–1066, 1968.

[Mal98] D. Mall. On holomorphic and transversely holomorphic foliations on Hopf surfaces. *J. reine angew. Math.*, 501:41–69, 1998.

[Mol88] P. Molino. *Riemannian Foliations*, volume 73 of *Progress in Math*. Birkhäuser, Basel, 1988.

[Pic90] P. Piccinni. On some classes of 2-dimensional Hermitian manifolds. *J. Math. Pures et Appl.*, 69:227–237, 1990.

[PPS93] H. Pedersen, Y. S. Poon, and A. Swann. The Einstein-Weyl equations in complex and quaternionic geometry. *Differential Geometry and its Applications*, 3(4):309–322, 1993.

[Tri82] F. Tricerri. Some examples of locally conformal Kähler manifolds. *Rend. Semin. Mat. Univ. Politecn. Torino*, 40:81–92, 1982.

[Vai76] I. Vaisman. On locally conformal almost Kähler manifolds. *Israel J. Math.*, 24:338–351, 1976.

[Vai79] I. Vaisman. Locally conformal Kähler manifolds with parallel Lee form. *Rend. Mat. Roma*, 12:263–284, 1979.

[Vai82] I. Vaisman. Generalized Hopf manifolds. *Geometriae Dedicata*, 13:231–255, 1982.

Maurizio Parton, parton@dm.unipi.it,
Dipartimento di Matematica “Leonida Tonelli”,
via Filippo Buonarroti 2, I-56127 Pisa,
Italy.